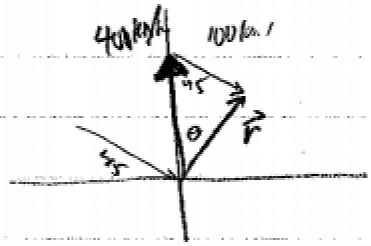
MCV4U – Exam Review Day 2

1. An airplane heads due north with a velocity of 400 km/h and encounters a wind of 100 km/h from the northeast. Determine the resultant velocity of the airplane.



$|\vec{r}|^2 = 400^2 + 100^2 - 2(400)(100)\cos 45$

$|\vec{r}| = 336.8$

$\dfrac{100}{\sin\theta} = \dfrac{336.8}{\sin 45}$

$\theta = \sin^{-1}\left(\dfrac{100\sin 45}{336.8}\right) \doteq 12.1$

∴ The plane is travelling 336.8 km [N 12.1 W].

2. If $\vec{a}$ and $\vec{b}$ are unit vectors and the angle between them is 60°, calculate $(6\vec{a}+\vec{b})\bullet(\vec{a}-2\vec{b})$.

$(6\vec{a} + \vec{b})\cdot(\vec{a} - 2\vec{b})$

$= 6|\vec{a}|^2 - 12\vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{b} - 2|\vec{b}|^2$

$= 6(1)^2 - 11\vec{a}\cdot\vec{b} - 2(1)^2$

$= 4 - 11|\vec{a}||\vec{b}|\cos 60$

$= 4 - 11(1)(1)\frac{1}{2}$

$= -\frac{3}{2}$

3. Given $\vec{a} = [6,2,-3]$, determine the Cartesian form of a unit vector in the opposite direction as $\vec{a}$.

$\vec{a} = [6, 2, -3]$

$|\vec{a}| = \sqrt{49} = 7$

$\therefore -\frac{1}{7}[6,2,-3] = [-\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}]$

4. Triangle PQR has vertices P(4,3), Q(-8,-7) R(6,-8),
Determine the vector equation of the line that passes through R and orthogonal to QP.

$\vec{QP} = [12, 10]$

$\therefore \vec{d} = [-10, -12]$

$= -2[5,-6]$

$\therefore \vec{d} = [5,-6]$

$\therefore \vec{r} = [6,-8] + t[5,-6],\ t \in \mathbb{R}$.

5. Given the points A(1,3,-2), B(0,3,2), C(1,1,-3), and vectors $\vec{u}=[-2,1,3], \vec{v}=[0,3,2], \vec{w}=[-3,1,2], \vec{x}=[6,-2,m+2]$, determine

a) $|\overrightarrow{AB}|$

b) $\overrightarrow{AB}\bullet\overrightarrow{BC}$

c) the acute angle between $\vec{u}\ \&\ \vec{v}$

d) the value of m such that $\vec{w}\ \&\ \vec{x}$ are collinear

e) the area of triangle ABC

f) $proj_{\vec{w}}\vec{u}$

a) $\overrightarrow{AB}=[-1,0,4]$

$|\overrightarrow{AB}|=\sqrt{17}$

b) $\overrightarrow{AB}\cdot\overrightarrow{BC}=[-1,0,4]\cdot[1,-2,-5]$

$=-1+0-20$

$=-21$

c) $\cos\theta=\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

$\cos\theta=\dfrac{[-2,1,3]\cdot[0,3,2]}{\sqrt{14}\ \sqrt{13}}$

$\theta=\cos^{-1}\left(\dfrac{9}{\sqrt{182}}\right)=48^\circ$

d) $\vec{w}=k\vec{x}$ (if $\vec{w}\ \&\ \vec{x}$ are collinear)

$[-3,1,2]=k[6,-2,m+2]$

$-3=6k$ $\quad 1=-2k$ $\quad 2=(m+2)k$

$-\frac{1}{2}=k$ $\quad -\frac{1}{2}=k$ $\quad 2=(m+2)(-\frac{1}{2})$

$-4=m+2$

$\boxed{-6=m}$

e) $\overrightarrow{AB}=[-1,0,4]$

$\overrightarrow{BC}=[1,-2,-5]$

$$\begin{array}{cc} -1 & 1 \\ 0 & -2 \\ 4 & -5 \\ -1 & 1 \\ 0 & -2 \\ 4 & -5 \end{array}$$

$A_{parallelog}=|\overrightarrow{AB}\times\overrightarrow{BC}|$

$A_\Delta=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{BC}|$

$=\frac{1}{2}|[8,-1,2]|$

$=\frac{1}{2}(\sqrt{69})$

$=\dfrac{\sqrt{69}}{2}$

f) $proj_{\vec{w}}\vec{u}=\dfrac{\vec{u}\cdot\vec{w}}{|\vec{u}|^2}\vec{u}=\dfrac{[-2,1,3]\cdot[-3,1,2]}{14}[-2,1,3]$

$=\dfrac{6+1+6}{14}[-2,1,3]$

$=\dfrac{13}{14}[-2,1,3]$

$=\left[\dfrac{-13}{7},\dfrac{13}{14},\dfrac{39}{14}\right]$

6. The vectors $\vec{p} = [-4, p, -2]$ and $\vec{q} = [-2, 3, 6]$ are oriented such that $\cos\theta = \dfrac{4}{21}$ where θ is the angle between the vectors. Determine the value(s) of p.

$$\vec{p}\cdot\vec{q} = |\vec{p}||\vec{q}|\cos\theta$$

$$[-4,p,-2]\cdot[-2,3,6] = \left(\sqrt{p^2+20}\right)\sqrt{49}\cos\theta$$

$$8 + 3p - 12 = 7\cdot\frac{4}{21}\sqrt{p^2+20}$$

$$3p - 4 = \frac{4}{3}\sqrt{p^2+20}$$

$$9p - 12 = 4\sqrt{p^2+20}$$

$$81p^2 - 216p + 144 = 16(p^2+20)$$

$$65p^2 - 216p - 176 = 0$$

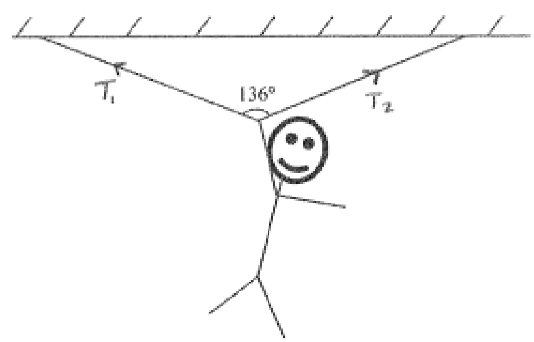
$\therefore$ p can be 4 or $-\frac{44}{65}$

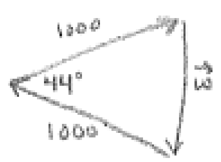
$$p = \frac{216 \pm 304}{130}$$

$$\text{so } p = \frac{216+304}{130} \quad \text{or} \quad p = \frac{216-304}{130}$$

$$p = 4 \qquad p = \frac{-44}{65}$$

7. In my spare time, I like to hang from cables as shown.
If the tension in each cable is 1000N, how much do I weigh?

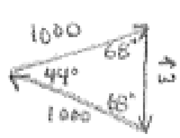




$$|\vec{w}|^2 = 1000^2 + 1000^2 - 2(1000)(1000)\cos 44°$$

$$\boxed{|\vec{w}| = 749 \text{ N}}$$

* Note: ~~since~~ because tensions are equal, there would be symmetry and could also solve with sine law.



$$\frac{|\vec{w}|}{\sin 44°} = \frac{1000}{\sin 68°}$$

8. Given the line $\vec{r} = [0,-5,7] + t[-2,0,3]$, write a parametric equation for the line

$x = 0 - 2t = -2t$
$y = -5 + 0t = -5$
$z = 7 + 3t$

9. State an equation of a line that

a) Is parallel to $\vec{r} = [2,0,7] + t[-2,5,3]$, and has an x-intercept of -7

same direction

pt (-7, 0, 0)

$\vec{r} = [-7, 0, 0] + t[-2, 5, 3]$

10. Given the plane 3x + 2y – 5z – 7 = 0,

a) State any point on the plane

Anything that satisfies equation, such as

(1, 2, 0) or (0, 1, -1), or...

b) Is the line $\vec{r} = [1,-3,-2] + t[-2,4,-1]$ on the plane?

(1, -3, -2) is on plane

if $[-2, 4, -1] \perp [3, 2, -5]$ entire line is on plane

$[-2, 4, -1] \cdot [3, 2, -5] = 7$ ∴ Not on plane

11. Given the vectors $\vec{a}$ = [4, 1, -4] and $\vec{b}$ = [2, -3, 5], find two vectors that are perpendicular to both $\vec{a}$ and $\vec{b}$?

$\vec{a} \times \vec{b}$ is $\perp$ to both.

$$\begin{matrix} 4 & 1 & -4 & 4 & 1 & -4 \\ 2 & -3 & 5 & 2 & -3 & 5 \end{matrix}$$

$\vec{a} \times \vec{b} = [5 - 12, -8 - 20, -12 - 2]$

$= \boxed{[-7, -28, -14]}$

second answer is any scalar multiple, such as $\boxed{[1, 4, 2]}$