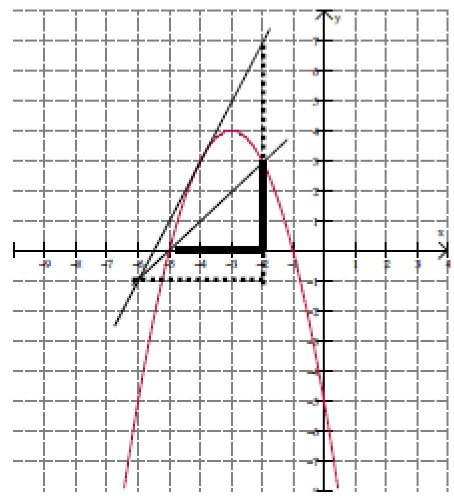
MCV4U – Exam Review Day 4

1. Find the following rates of change using a graphical method (using the graph at the right).
   a) The average rate of change from x = -5 to x = -2.
   b) The instantaneous rate of change at x = -4.



a) $Rate_{x=-5 \text{ to } x=-2} = \frac{3}{3} = 1$

b) $Rate_{x=-4} = \frac{8}{4} = 2$

2. Using the first principles definition of the derivative find
   a) $f'(x)$ if $f(x) = 2x^2 + 7x - 3$
   b) $h'(x)$ if $h(x) = 9^x$
   c) $g'(-1)$ if $g(x) = -\frac{5}{2x+3}$

a) $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{2(x+h)^2 + 7(x+h) - 3 - [2x^2 + 7x - 3]}{h}$

$f'(x) = \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 + 7x + 7h - 3 - 2x^2 - 7x + 3}{h}$

$f'(x) = \lim_{h\to 0} \frac{4xh + 2h^2 + 7h}{h}$

$f'(x) = \lim_{h\to 0} (4x + 2h + 7)$

$\therefore f'(x) = 4x + 7$

b) $h'(x) = \lim_{s\to 0} \frac{h(x+s) - h(x)}{s}$

$h'(x) = \lim_{s\to 0} \frac{9^{x+s} - 9^x}{s}$

$h'(x) = 9^x \lim_{s\to 0} \frac{9^s - 1}{s}$

$\therefore h'(x) = 9^x \ln(9)$

c) $g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$

$g'(x) = \lim_{h\to 0} \frac{-\frac{5}{2(x+h)+3} - \left(-\frac{5}{2x+3}\right)}{h}$

$g'(x) = \lim_{h\to 0} \frac{-\frac{5}{2x+2h+3} + \frac{5}{2x+3}}{h}$

$g'(x) = \lim_{h\to 0} \frac{\frac{-5(2x+3)}{(2x+2h+3)(2x+3)} + \frac{5(2x+2h+3)}{(2x+3)(2x+2h+3)}}{h}$

$g'(x) = \lim_{h\to 0} \frac{\frac{-10x - 15 + 10x + 10h + 15}{(2x+2h+3)(2x+3)}}{h}$

$g'(x) = \lim_{h\to 0} \frac{\frac{10h}{(2x+2h+3)(2x+3)}}{h}$

$g'(x) = \lim_{h\to 0} \frac{10h}{(2x+2h+3)(2x+3)} \times \frac{1}{h}$

$g'(x) = \lim_{h\to 0} \frac{10}{(2x+2h+3)(2x+3)}$

$\therefore g'(x) = \frac{10}{(2x+3)^2}$

3. A certain bacteria culture starts with a population of 120. The population grew to 5000 in 1 day.

a) Find a formula for the number of bacteria after t hours.

b) How many bacteria were there after 14 hours?

a) $P(t)=120e^{kt}$

$5000=120e^{k24}$

$\frac{125}{3}=e^{24k}$

$24k=\ln\left(\frac{125}{3}\right)$

$k=\frac{\ln\left(\frac{125}{3}\right)}{24}\doteq 0.1554$

The formula is $P(t)=120e^{0.1554t}$

b) $P(t)=120e^{0.1554t}$

$P(14)=120e^{0.1554(14)}$

$P(14)=1056.896$

After 14 hours there were about 1057 bacteria.

4. A transportation company sells 2000 bus tickets per day when the price is $10 per ticket. They believe that for every $1 increase in ticket price they will sell 100 tickets less. Use the calculus to determine what price the company should charge to maximize their revenue.

Let x represent the number of $1 increases in ticket price and R(x) be the revenue as a function of the number of increases.

$R(x)=(10+x)(2000-100x)$

$R(x)=20000-1000x+2000x-100x^2$

$R(x)=20000+1000x-100x^2$

$R'(x)=1000-200x$

$1000-200x=0$ $\left[\text{The max occurs where } R'(x)=0\right]$

$200x=1000$

$\therefore x=5$

If x = 5, then the price is 10 + 5 = $15.00 per ticket. This will yield the highest possible revenue.

5. A sample of radioactive Plutonium started with a mass of 16.2 mg. This isotope's half-life is 2.7 hours.

a) Find a formula for the amount of Plutonium remaining after t hours.

b) How long (from the original sample) will it take for the sample size to be down to only 1 mg?

a) $N(t)=16.2e^{kt}$

$8.1=16.2e^{k(2.7)}$

$\frac{8.1}{16.2}=\frac{16.2e^{2.7k}}{16.2}$

$0.5=e^{2.7k}$

$2.7k=\ln 0.5$

$k=\frac{\ln 0.5}{2.7}\doteq -0.2567$

Formula: $N(t)=16.2e^{-0.2567t}$

b) $N(t)=16.2e^{-0.2567t}$

$1=16.2e^{-0.2567t}$

$\frac{1}{16.2}=\frac{16.2e^{-0.2567t}}{16.2}$

$\frac{1}{16.2}=e^{-0.2567t}$

$-0.2567t=\ln\left(\frac{1}{16.2}\right)$

$t=-\frac{\ln\left(\frac{1}{16.2}\right)}{0.2567}$

$\therefore t\doteq 10.85$ hours

6. Sketch the graph of the following functions using local maximum/minimums, intercepts, asymptotes, points of inflection and concavity.

a) $f(x) = x^4 - 8x^3 + 18x^2$

Intercepts: $x^4 - 8x^3 + 18x^2 = 0$

$x^2(x^2 - 8x + 18) = 0$

$x^2 = 0$ or $x^2 - 8x + 18 = 0$

$\therefore x = 0$ $\quad x^2 - 8x + 18 = 0$ has no solution

$f'(x) = 4x^3 - 24x^2 + 36x$

$4x^3 - 24x^2 + 36x = 0$

$4x(x^2 - 6x + 9) = 0$

$4x = 0 \quad x^2 - 6x + 9 = 0$

$x = 0 \quad (x-3)^2 = 0$

$x = 3$

$f'(x) = 4x^3 - 24x^2 + 36x$

$f''(x) = 12x^2 - 48x + 36$

$12x^2 - 48x + 36 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$\therefore x = 3, 1$ [possible POI]

$f(1) = (1)^4 - 8(1)^3 + 18(1)^2$

$f(1) = 11$

$f(3) = (3)^4 - 8(3)^3 + 18(3)^2$

$f(3) = 81 - 216 + 162$

$f(3) = 27$

$f''(0) = 12(0)^2 - 48(0) + 36 = 36$

$\therefore x = 0$ is a local min (concave up)

$f''(2) = 12(2)^2 - 48(2) + 36 = -12$

concave down

$f''(3) = 12(3)^2 - 48(3) + 36 = 0$

$f''(4) = 12(4)^2 - 48(4) + 36 = 36$

$x = 3$ is a POI since $f''(x)$ changes sign

Summary:

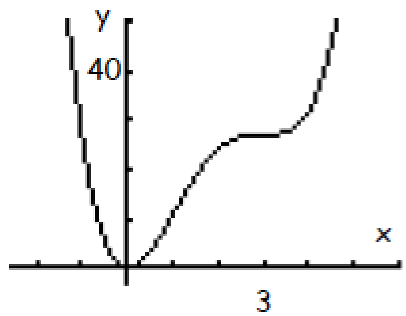
Local min at $(0,0)$

X-intercept at $(0,0)$

POI at $(3,27)$ & $(1,11)$

Concave Down: $1 < x < 3$

Concave Up: $x < 1$ & $x > 3$



b) $f(x)=\frac{10}{(x+2)^2}$

y-intercept:

$f(0)=\frac{10}{(0+2)^2}$

$f(0)=\frac{10}{4}$

$f(0)=2.5$

No x-intercept since

$\frac{10}{(x+2)^2}\neq 0$

Vertical Asymptote at $x=-2$

Since $f(x)=\frac{10}{(x+2)^2}>0$, the graph goes up on both sides of the asymptote.

$f(x)=10(x+2)^{-2}$

$f'(x)=-20(x+2)^{-3}$

$f'(x)=-\frac{20}{(x+2)^3}$

Since $-\frac{20}{(x+2)^3}\neq 0$

No local max/min.

$f'(x)=-20(x+2)^{-3}$

$f''(x)=60(x+2)^{-4}$

$f''(x)=\frac{60}{(x+2)^4}$

Since $\frac{60}{(x+2)^4}>0$, the graph is entirely concave up.

Summary:

Local min/max: none

y-intercept at $(0,2.5)$

POI: none

Concave Up: $x<-2$ & $x>-2$