UNIT 7: LINES & PLANES

Communication reasoning, in writing and use of mathematical language, symbols and conventions will be assessed throughout this test.

PLEASE CHOOSE ONLY 2 OUT OF THE 3 QUESTIONS FROM BELOW. CIRCLE ON THIS PAGE WHICH QUESTIONS YOU CHOSE TO DO.

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1. Determine the Cartesian equation of the plane that is parallel to the line with equation $x = -2y = 3z$ and that contains the line of intersection of the planes with equations $x - y + z = 1$ and $2y - z = 0$.

$x = t$
$y = -\frac{1}{2}t$
$z = \frac{1}{3}t$

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2. Determine the solution to the following system of equations:

① $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0$

② $\frac{2}{a} + \frac{3}{b} + \frac{2}{c} = \frac{13}{6}$

③ $\frac{4}{a} - \frac{2}{b} + \frac{3}{c} = \frac{5}{2}$

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3. The line $\vec{r} = (-8, -6, -1) + s(2, 2, 1)$, $s \in \mathbf{R}$, intersects the xz- and yz-coordinate planes at the points A and B, respectively. Determine the length of line segment AB.

	LEVEL 1	LEVEL 2	LEVEL 3	LEVEL 4
TI	Uses planning skills with limited effectiveness.	Uses planning skills with some effectiveness.	Uses planning skills with considerable effectiveness.	Uses planning skills with high degree of effectiveness.
TI	Uses critical / creative thinking process with limited effectiveness.	Uses critical / creative thinking process with some effectiveness.	Uses critical / creative thinking process with considerable effectiveness.	Uses critical / creative thinking process with a high degree of effectiveness.
CO	Communicates for different audiences and purposes with limited effectiveness.	Communicates for different audiences and purposes with some effectiveness.	Communicates for different audiences and purposes with considerable effectiveness.	Communicates for different audiences and purposes with a high degree of effectiveness.
CO	Expresses and organizes mathematical thinking with limited effectiveness.	Expresses and organizes mathematical thinking with some effectiveness.	Expresses and organizes mathematical thinking with considerable effectiveness.	Expresses and organizes mathematical thinking with a high degree of effectiveness.

1. Determine the Cartesian equation of the plane that is parallel to the line with equation ① $x = -2y = 3z$ and that contains the line of intersection of the planes with equations ② $x - y + z = 1$ and ③ $2y - z = 0$.

SOLUTION:

~~The line with equation $x = -2y = 3z$ has parametric equations: $x = s, y = -\frac{1}{2}s, z = \frac{1}{3}s, s \in \mathbf{R}$. This has the equivalent vector form:~~

~~$\vec{r} = s\left(1, -\frac{1}{2}, \frac{1}{3}\right), s \in \mathbf{R}$.~~

~~The line of intersection of the two planes ① $x - y + z = 1$ and ② $2y - z = 0$ is:~~

~~$y = \frac{1}{2}z$~~

~~$x - \frac{1}{2}z + z = 1$~~

~~$x = 1 - \frac{1}{2}z$~~

~~$x = 1 - \frac{1}{2}t, y = \frac{1}{2}t, z = t, t \in \mathbf{R}$. Which has a vector equation of:~~

~~$\vec{r} = (1, 0, 0) + t\left(-\frac{1}{2}, \frac{1}{2}, 1\right), t \in \mathbf{R}$. The vector equation of the plane with the given properties is thus:~~

~~$\vec{r} = (1, 0, 0) + t\left(-\frac{1}{2}, \frac{1}{2}, 1\right) + s\left(1, -\frac{1}{2}, \frac{1}{3}\right), s, t \in \mathbf{R}$.~~

~~The normal vector for the plane is then:~~

~~$\left(-\frac{1}{2}, \frac{1}{2}, 1\right) \times \left(1, -\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2} \cdot \frac{1}{3} - \left(1 \cdot -\frac{1}{2}\right),\right.$~~

~~$\left.1 \cdot 1 - \left(-\frac{1}{2} \cdot \frac{1}{3}\right), -\frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2} \cdot 1\right) = \left(\frac{2}{3}, \frac{7}{6}, -\frac{1}{4}\right)$.~~

~~Or equivalently $(8, 14, -3)$. The Cartesian equation is then: $8x + 14y - 3z + D = 0$, and must contain the point $(1, 0, 0)$.~~

~~$8(1) + D = 0$~~

~~$D = -8$.~~

~~$8x + 14y - 3z - 8 = 0$.~~

$Ax + By + Cz + D = 0$

$8x + 14y - 3z + D = 0$

sub $(1, 0, 0)$

$8(1) + 14(0) - 3(0) + D = 0$

$D = -8$

$8x + 14y - 3z - 8 = 0$

Out of 10

$\boxed{x = -2y = 3z = s}$ symmetric equations

① $x = s$, $y = -\frac{1}{2}s$, $z = \frac{1}{3}s$ = parametric equations

② $x - y + z = 1$

③ $2y - z = 0$

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③ $2y - z = 0$

$y = \frac{1}{2}z$

Let $z = s$

$y = \frac{1}{2}s$ & $z = s$

sub y & z into ②

$x - (\frac{1}{2}s) + (s) = 1$

$x = 1 - \frac{1}{2}s$

④ $x = 1 - \frac{1}{2}s$, $y = \frac{1}{2}s$, $z = s$

from ④ $\vec{m}_2 = (\frac{-1}{2}, \frac{1}{2}, 1)$ & $P_0 = (1, 0, 0)$

from ① $\vec{m}_1 = (1, -\frac{1}{2}, \frac{1}{3})$

$\therefore [x, y, z] = (1, 0, 0) + s(\frac{-1}{2}, \frac{1}{2}, 1) + t(1, \frac{-1}{2}, \frac{1}{3})$

vector equation

$\vec{m}_1 \times \vec{m}_2 = (\frac{2}{3}, \frac{7}{6}, \frac{-1}{4})$ or $(8, 14, -3)$

$\therefore \vec{n} = (8, 14, -3)$

2. Determine the solution to the following system of equations:

① $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0$

② $\frac{2}{a} + \frac{3}{b} + \frac{2}{c} = \frac{13}{6}$

③ $\frac{4}{a} - \frac{2}{b} + \frac{3}{c} = \frac{5}{2}$

Out of 10

SOLUTION:

~~① $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0$~~

~~② $\frac{2}{a} + \frac{3}{b} + \frac{2}{c} = \frac{13}{6}$~~

~~③ $\frac{4}{a} - \frac{2}{b} + \frac{3}{c} = \frac{5}{2}$~~

~~Equation ② − 2 × equation ①:~~

~~$\frac{1}{b} + \frac{4}{c} = \frac{13}{6}$ = equation ④~~

~~Equation ③ − 4 × equation ①: $-\frac{6}{b} + \frac{7}{c}$~~

~~$m_3 = \vec{n} \times \vec{n_1} = (-1, 0, 1) = \frac{5}{2}$ = equation ⑤~~

~~Equation ⑤ + 6 × equation ④:~~

~~$\frac{31}{c} = 15.5$ or $c = 2$~~

~~Substituting $c = 2$ into equation ④:~~

~~$\frac{1}{b} + 2 = \frac{13}{6}$ or $b = 6$~~

~~Substituting $c = 2$ and $b = 6$ into equation ①:~~

~~$\frac{1}{a} + \frac{1}{6} - \frac{1}{2} = 0$ or $a = 3$~~

~~(3, 6, 2)~~

$-2 \times ① \rightarrow -\frac{2}{a} - \frac{2}{b} + \frac{2}{c} = 0$

② $+ \quad \frac{2}{a} + \frac{3}{b} + \frac{2}{c} = \frac{13}{6}$

④ $\frac{1}{b} + \frac{4}{c} = \frac{13}{6}$ ✓

$-4 \times ① \rightarrow -\frac{4}{a} - \frac{4}{b} + \frac{4}{c} = 0$

$+ \quad \frac{4}{a} - \frac{2}{b} + \frac{3}{c} = \frac{5}{2}$ ✓

⑤ $-\frac{6}{b} + \frac{7}{c} = \frac{5}{2}$ ✓

$6 \times ④ \rightarrow \frac{6}{b} + \frac{24}{c} = 13$

⑤ $\rightarrow -\frac{6}{b} + \frac{7}{c} = \frac{5}{2}$ ✓

$+$

$\frac{31}{c} = \frac{31}{2}$

$31c = 62$

$c = 2$ ✓

sub $c = 2$ into ④

$\frac{1}{b} + \frac{4}{2} = \frac{13}{6}$

$\frac{1}{b} = \frac{1}{6}$ ✓

$b = 6$

sub $c = 2$ & $b = 6$ into ①

$\frac{1}{a} + \frac{1}{6} - \frac{1}{2} = 0$

$\frac{1}{a} = \frac{1}{3}$ ✓

$a = 3$ ✓

$\therefore (a, b, c) = (3, 6, 2)$ ✓

3. The line $\vec{r} = (-8, -6, -1) + s(2, 2, 1)$, $s \in \mathbf{R}$, intersects the xz- and yz-coordinate planes at the points A and B, respectively. Determine the length of line segment AB.

SOLUTION:

On the xz-plane, the point A has the coordinates $(x, 0, z)$ for any x, z. Similarly, on the yz-plane, the point B has the coordinates $(0, y, z)$ for any y, z. Now the task is to find the required values of s for these points. Starting with the x component of point B, we have $0 = -8 + 2s$ or $s = 4$. So point B is $(-8, -6, -1) + 4(2, 2, 1) = (0, 2, 3)$. For point A, we need the y coordinate to equal 0. So $0 = -6 + 2s$ or $s = 3$. So point A is $(-8, -6, -1) + 3(2, 2, 1) = (-2, 0, 2)$.
Now we need to find the distance

$$d = \sqrt{(0 - (-2))^2 + (2 - 0)^2 + (3 - 2)^2}$$
$$= \sqrt{4 + 4 + 1}$$
$$= \sqrt{9}$$
$$= 3$$

parametric

$$\boxed{\begin{aligned} x &= -8 + 2s \\ y &= -6 + 2s \\ z &= -1 + s \end{aligned}}$$

Out of 10

yz-plane (point) $\rightarrow$ let $x = 0$

$0 = -8 + 2s$

$4 = s$

sub $s = 4$ into $\vec{r}$

$\vec{r} = (-8, -6, -1) + (4)(2, 2, 1)$

$\vec{r} = (0, 2, 3) \rightarrow \underline{A}$

xz-plane (point) $\rightarrow$ let $y = 0$

$0 = -6 + 2s$

$3 = s$

sub $s = 3$ into $\vec{r}$

$\vec{r} = (-8, -6, -1) + (3)(2, 2, 1)$

$\vec{r} = (-2, 0, 2) \rightarrow \underline{B}$

$\vec{AB} = (-2, 0, 2) - (0, 2, 3)$

$\vec{AB} = (-2, -2, -1)$

$|\vec{AB}| = \sqrt{(-2)^2 + (-2)^2 + (-1)^2}$

$|\vec{AB}| = 3$